

# Self-Dual Codes over $\mathbb{Z}_2 \times (\mathbb{Z}_2 + u\mathbb{Z}_2)$

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## Abstract

In this paper, we study self-dual codes over  $\mathbb{Z}_2 \times (\mathbb{Z}_2 + u\mathbb{Z}_2)$ , where  $u^2 = 0$ . Three types of self-dual codes are defined. For each type, the possible values  $\alpha, \beta$  such that there exists a code  $\mathcal{C} \subseteq \mathbb{Z}_2^\alpha \times (\mathbb{Z}_2 + u\mathbb{Z}_2)^\beta$  are established. We also present several approaches to construct self-dual codes over  $\mathbb{Z}_2 \times (\mathbb{Z}_2 + u\mathbb{Z}_2)$ . Moreover, the structure of two-weight self-dual codes is completely obtained for  $\alpha \cdot \beta \neq 0$ .

**Key Words** Linear code, Self-dual codes, Two-weight self-dual codes

## 1 Introduction

A binary code  $C$  of length  $n$  over the finite field  $\mathbb{Z}_2$  is a subset of  $\mathbb{Z}_2^n$ . If  $C$  is a vector subspace of  $\mathbb{Z}_2^n$ , then we call this code is linear. A quaternary code  $\mathcal{C}$  is a subset of  $\mathbb{Z}_4^n$  and it is said to be linear if it is a submodule. In [1], Delsarte proposed the definition of additive codes, which are subgroups of the underlying abelian group in a translation association scheme. In particular, a binary Hamming scheme which is the only structures for the abelian group are those of the form  $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$  when the underlying abelian group is of order  $2^n$ , where  $n = \alpha + 2\beta([2])$ . Hence, the only

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additive codes in a binary Hamming scheme are the subgroups  $\mathcal{C}$  of  $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ . In order to distinguish them from additive codes over finite fields (see [3–6]), we call them  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes. Foundational results on  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes, including the generator matrix, the existence and the construction of self-dual codes, can be found in [7].

Now, we present another important ring  $R$  which contains four elements, where  $R = \mathbb{Z}_2 + u\mathbb{Z}_2 = \{0, 1, u, 1 + u\}$  with  $u^2 = 0$ . It is well known that the ring  $\mathbb{Z}_2$  is a subring of the ring  $R$ . Similar as that  $\mathbb{Z}_2\mathbb{Z}_4$ -additive codes, we define the following set:

$$\mathbb{Z}_2^\alpha \times R^\beta = \{(\mathbf{a}, \mathbf{b}) \mid \mathbf{a} \in \mathbb{Z}_2^\alpha, \mathbf{b} \in R^\beta\}.$$

We cannot directly define an algebraic structure endowing the set  $\mathbb{Z}_2R$ , which is not well defined with respect to the usual multiplication by  $u \in R$ . So this set is not  $R$ -module. In order to make it well defined and enrich with an algebraic structure we introduce a new multiplication as follows.

Define a map

$$\eta : R \longrightarrow \mathbb{Z}_2, \eta(r + uq) \longmapsto r,$$

Clearly, the map  $\eta$  is a ring homomorphism. Using this map, we can define a scalar multiplication as follows: for  $\nu = (a_1, a_2, \dots, a_\alpha \mid b_1, b_2, \dots, b_\beta) \in \mathbb{Z}_2^\alpha \times R^\beta$  and  $d \in R$ , we have

$$d\nu = (\eta(d)a_1, \dots, \eta(d)a_\alpha \mid db_1, \dots, db_\beta). \quad (1.1)$$

**Definition 1.1.** A linear code  $\mathcal{C}$  is called a  $\mathbb{Z}_2R$  linear codes if it is a  $R$ -submodule of  $\mathbb{Z}_2^\alpha \times R^\beta$  with respect to the scalar multiplication defined in (1.1). Then the binary image of  $\Phi(\mathcal{C}) = \mathcal{C}$  is called a  $\mathbb{Z}_2R$ -linear code of length  $n = \alpha + 2\beta$ , where  $\Phi$  is a map from  $\mathbb{Z}_2^\alpha \times R^\beta$  to  $\mathbb{Z}_2^n$  defined as

$$\Phi(a, b) = (a_1, \dots, a_\alpha \mid \phi(b_1), \dots, \phi(b_\beta)),$$

for all  $a = (a_1, \dots, a_\alpha) \in \mathbb{Z}_2^\alpha$ , and  $b = (b_1, \dots, b_\beta) \in R^\beta$ . Furthermore,  $\phi : R$  to  $\mathbb{Z}_2$  is defined by  $\phi(0) = (0, 0)$ ,  $\phi(1) = (0, 1)$ ,  $\phi(u) = (1, 1)$ ,  $\phi(1 + u) = (1, 0)$ .

With the above preparation, Aydogdu et al. in [8] obtained the standard form matrix of  $\mathbb{Z}_2R$  linear codes.

**Theorem 1.2.** [8] Let  $\mathcal{C}$  be a  $\mathbb{Z}_2R$  linear code of type  $(\alpha, \beta; \gamma, \delta; \kappa)$ . Then  $\mathcal{C}$  is permutation equivalent to a  $\mathbb{Z}_2R$  linear code with the standard form matrix

$$G = \left( \begin{array}{cc|ccc} I_\kappa & A_1 & uT & 0 & 0 \\ 0 & 0 & uD & uI_{\gamma-\kappa} & 0 \\ 0 & S & B_1 + uB_2 & A & I_\delta \end{array} \right), \quad (1.2)$$

where  $A, A_1, B_1, B_2, D, S$  and  $T$  are matrices over  $\mathbb{Z}_2$ .

From Theorem 1.2, it is easy that see  $\mathbb{Z}_2R$  linear code is isomorphic to  $\mathbb{Z}_2^\gamma \times \mathbb{Z}_2^{2\delta}$ , and it has  $|\mathcal{C}| = 2^{\gamma+2\delta}$ . Moreover, having the generator matrix as above, we say that  $\mathcal{C}$  is of type  $(\alpha, \beta; \gamma, \delta; \kappa)$ . Also,  $\kappa$  can be defined as follows:

Let  $X$  (respectively  $Y$ ) be the set of  $\mathbb{Z}_2$  (respectively  $R$ ) coordinates positions, hence  $|X| = \alpha$  (respectively  $|Y| = \beta$ ). Unless otherwise stated, the  $X$  corresponds to the first  $\alpha$  coordinates and  $Y$  corresponds to the last  $\beta$  coordinates. Call  $\mathcal{C}_X$  (respectively  $\mathcal{C}_Y$ ) the punctured code of  $\mathcal{C}$  by deleting the coordinates outside  $X$  (respectively  $Y$ ). Let  $\mathcal{C}_b$  be the subcode of  $\mathcal{C}$  which contains all codewords having the form of  $(x|y_1, y_2, \dots, y_\beta)$ , where  $x \in \mathbb{Z}_2^\alpha, y_i \in \{0, u\}, i = 1, 2, \dots, \beta$ . Then  $\kappa = \dim(\mathcal{C}_b)_X$ . For the case  $\alpha = 0$ , we will take  $\kappa = 0$ .

**Definition 1.3.** An inner product for two vectors  $\mathbf{v} = (v_1, \dots, v_\alpha | v_{\alpha+1}, \dots, v_{\alpha+\beta}), \mathbf{w} = (w_1, \dots, w_\alpha | w_{\alpha+1}, \dots, w_{\alpha+\beta}) \in \mathbb{Z}_2^\alpha \times (\mathbb{Z}_2 + u\mathbb{Z}_2)^\beta$  is defined as

$$\langle \mathbf{v}, \mathbf{w} \rangle = u \left( \sum_{i=1}^{\alpha} v_i w_i \right) + \sum_{j=\alpha+1}^{\alpha+\beta} v_j w_j \in R = \mathbb{Z}_2 + u\mathbb{Z}_2. \quad (1.3)$$

Hence, we have the definition of dual codes.

**Definition 1.4.** Let  $\mathcal{C}$  be a  $\mathbb{Z}_2R$  code. We denote the dual of  $\mathcal{C}$  by  $\mathcal{C}^\perp$ , which is defined as

$$\mathcal{C}^\perp = \{ \mathbf{w} \in \mathbb{Z}_2^\alpha \times (\mathbb{Z}_2 + u\mathbb{Z}_2)^\beta \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0 \text{ for all } \mathbf{v} \in \mathcal{C} \}.$$

We say that  $\mathcal{C}$  is self-orthogonal if and only if  $\mathcal{C} \subseteq \mathcal{C}^\perp$  and  $\mathcal{C}$  is self-dual if and only if  $\mathcal{C} = \mathcal{C}^\perp$ .

Following above definitions, the standard form generator matrix of  $\mathcal{C}^\perp$  can be obtained.

**Theorem 1.5.** [8] Let  $\mathcal{C}$  be a  $\mathbb{Z}_2R$  linear code of type  $(\alpha, \beta; \gamma, \delta; \kappa)$  with standard form matrix defined in (1.2). Then the generator matrix of  $\mathcal{C}^\perp$  is given as

$$H = \left( \begin{array}{cc|cc} A_1^t & I_{\alpha-\kappa} & 0 & 0 \\ 0 & 0 & 0 & uI_{\gamma-\kappa} \\ T^t & 0 & I_{\beta+\kappa-\gamma-\delta} & D^t \end{array} \begin{array}{c} uS^t \\ uA^t \\ (B_1 + uB_2)^t + D^t A^t \end{array} \right),$$

where  $A, A_1, B_1, B_2, D$  and  $T$  are matrices over  $\mathbb{Z}_2$ .

From Theorem 1.5, we know that the dual code  $\mathcal{C}^\perp$  is a  $\mathbb{Z}_2R$  linear code of type  $(\alpha, \beta; \bar{\gamma}, \bar{\delta}; \bar{\kappa})$ , where

$$\begin{cases} \bar{\kappa} = \alpha - \kappa; \\ \bar{\gamma} = \alpha + \gamma - 2\kappa; \\ \bar{\delta} = \beta - \gamma - \delta + \kappa \end{cases}$$

Let  $(\mathbf{v}|\mathbf{w}) = (v_1, \dots, v_\alpha \mid w_1, \dots, w_\beta) \in \mathbb{Z}_2^\alpha \times R^\beta$ . The Gray map defined on  $R$  can be expressed as follows:  $\phi(a + bu) = (b, a + b)$ ,  $a, b \in \mathbb{Z}_2$  and the Lee weight is  $wt_L(a + bu) = wt_H(b, a + b)$ . Hence,  $wt_L(\mathbf{v}|\mathbf{w}) = wt_H(\mathbf{v}) + wt_L(\mathbf{w})$ .

Let  $\mathcal{C}$  be a  $\mathbb{Z}_2 R$  linear code of type  $(\alpha, \beta; \gamma, \delta; \kappa)$  and  $n = \alpha + 2\beta$ . The weight enumerator of code  $\mathcal{C}$  is defined as

$$W(X, Y) = \sum_{c \in \mathcal{C}} X^{n-wt_L(c)} Y^{wt_L(c)}.$$

Aydogdu et al. [8] gave the following result.

**Theorem 1.6.** [8] *Let  $\mathcal{C}$  be a  $\mathbb{Z}_2 R$  linear code. The relation between the weight enumerators of  $\mathcal{C}$  and  $\mathcal{C}^\perp$  is given by the following identity:*

$$W_{\mathcal{C}^\perp}(X, Y) = \frac{1}{|\mathcal{C}|} W_{\mathcal{C}}(X + Y, X - Y).$$

Based on above definitions, we can construct a self-dual code  $\mathcal{C}_1$  over  $\mathbb{Z}_2^4 R^2$  of type  $(4, 2; 2, 1; 2)$ , which is generated by the following matrix

$$\left( \begin{array}{cccc|cc} 1 & 0 & 1 & 0 & u & 0 \\ 0 & 1 & 0 & 1 & u & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array} \right).$$

It is easy to get the weight enumerator of  $\mathcal{C}_1$  is

$$W(X, Y) = X^8 + 14X^4Y^4 + Y^8,$$

which implies that all codewords have doubly-even weight.

In [9], Borges et al. studied self-dual codes over  $\mathbb{Z}_2 \mathbb{Z}_4$ . Three types of self-dual codes are defined. For each type, the possible values  $\alpha, \beta$  such that there exists a code  $\mathcal{C} \subseteq \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$  are established. Borges et al. defined a self-dual code is Type II if all the codewords have doubly-even weight. And they showed if  $\mathcal{C}$  is type II  $\mathbb{Z}_2 \mathbb{Z}_4$ -additive code, then  $\alpha \equiv 0 \pmod{8}$ . Recall that linear code  $\mathcal{C}_1$  over  $\mathbb{Z}_2^4 R^2$ , it is easy to find this case doesn't exist in [9]. Motivated by this discovery, we furthermore study self-dual codes over  $\mathbb{Z}_2 R$ . Similar as that in [9], three types of self-dual codes are defined. We also give the existence condition for each type and present several approaches to construct self-dual codes. Finally, we study self-dual codes over  $\mathbb{Z}_2 R$  with two nonzero weights, and the structure of these codes is described.

This paper is organized as follows. Section 2, we study the properties of self-dual codes over  $\mathbb{Z}_2 R$ , and give the existence conditions for three types. In Section 3, we give several approaches of constructing self-dual codes over  $\mathbb{Z}_2 R$ . Section 4, we determine the structure of two-weight self-dual codes over  $\mathbb{Z}_2 R$  for  $\alpha \cdot \beta \neq 0$ .

## 2 Self-dual codes over $\mathbb{Z}_2R$

**Lemma 2.1.** *Let  $\mathcal{C}$  be a  $\mathbb{Z}_2R$  linear self-dual code, then  $\mathcal{C}$  is of type  $(2\kappa, \beta; \beta + \kappa - 2\delta, \delta; \kappa)$ ,  $|\mathcal{C}| = 2^{\kappa+\beta}$  and  $\mathcal{C}_b = 2^{\kappa+\beta-\delta}$ .*

*Proof.* By Theorem 1.2 and Theorem 1.5, we finish the proof.  $\square$

Hence, we have the following result.

**Corollary 2.2.** *Let  $\mathcal{C}$  be a  $\mathbb{Z}_2R$  linear self-dual code of type  $(\alpha, \beta; \gamma, \delta; \kappa)$ , then  $\alpha$  and  $n$  are both even.*

**Lemma 2.3.** *Let  $\mathcal{C}$  be a  $\mathbb{Z}_2R$  linear self-dual code and  $(\mathbf{v}|\mathbf{w}) \in \mathcal{C}$ . Denote  $N(\mathbf{w})$  as the number of unit (1 or  $1+u$ ) coordinates of vector  $\mathbf{w} \in R^\beta$ , then  $wt_H(\mathbf{v})$  and  $N(\mathbf{w})$  are both even. Moreover, we have  $(\mathbf{1}^\alpha, \mathbf{0}^\beta)$ ,  $(\mathbf{0}^\alpha, \mathbf{u}^\beta)$  and  $(\mathbf{1}^\alpha, \mathbf{u}^\beta)$  are all in  $\mathcal{C}$ , where  $\mathbf{a}^r$  is defined as the tuple  $(\overbrace{a, a, \dots, a}^r)$ .*

*Proof.* Since  $\mathcal{C}$  is a self-dual code, then for any codeword  $(\mathbf{v} | \mathbf{w}) \in \mathcal{C}$ , we have  $\langle (\mathbf{v}|\mathbf{w}), (\mathbf{v}|\mathbf{w}) \rangle = u \cdot wt_H(\mathbf{v}) + N(\mathbf{w}) = 0 \in R$ . Note that  $wt_H(\mathbf{v})$  and  $N(\mathbf{w})$  are all integers, so  $wt_H(\mathbf{v})$  and  $N(\mathbf{w})$  are both even. Since  $wt_H(\mathbf{v})$  and  $N(\mathbf{w})$  are both even, then  $(\mathbf{1}^\alpha, \mathbf{0}^\beta)$ ,  $(\mathbf{0}^\alpha, \mathbf{u}^\beta)$  are in  $\mathcal{C}$ . We are done.  $\square$

**Lemma 2.4.** *Let  $\mathcal{C}$  be a linear self-dual code, then the subcode  $(\mathcal{C}_b)_X$  is a binary self-dual code.*

*Proof.* By Lemma 2.1, we have  $\mathcal{C}$  is of type  $(2\kappa, \beta; \beta + \kappa - 2\delta, \delta; \kappa)$ . Note that for any two codewords  $(\mathbf{x}|\mathbf{y}), (\mathbf{v}|\mathbf{w}) \in \mathcal{C}_b$ , one has  $\langle \mathbf{y}, \mathbf{w} \rangle = 0$ . This implies  $(\mathcal{C}_b)_X \subseteq (\mathcal{C}_b)_X^\perp$ . Since the dimension of  $(\mathcal{C}_b)_X$  is  $\kappa$  and the length of  $(\mathcal{C}_b)_X$  is  $\alpha = 2\kappa$ , we are done.  $\square$

**Lemma 2.5.** *Let  $\mathcal{C}$  be a linear self-dual code of type  $(2\kappa, \beta; \beta + \kappa - 2\delta, \delta; \kappa)$ . There exists an integer  $r$ ,  $1 \leq r \leq \kappa$ , such that each codeword in  $\mathcal{C}_Y$  appears  $2^r$  times in  $\mathcal{C}$  and  $|\mathcal{C}_Y| \geq 2^\beta$ .*

*Proof.* Denote the subcode  $\mathcal{C}_0 = \{(\mathbf{v}|\mathbf{0}) \in \mathcal{C}\}$ . It is easy to see that  $(\mathcal{C}_0)_X$  is a linear binary code with dimension  $r = \dim(\mathcal{C}_0)_X$ . Thus, any vector in  $\mathcal{C}_Y$  appears  $2^r$  times in  $\mathcal{C}$ . Note that  $(\mathcal{C}_0)_X \subseteq (\mathcal{C}_b)_X$ , then  $r \leq \kappa$ . Since  $|\mathcal{C}| = 2^{\beta+\kappa} = |\mathcal{C}_Y||\mathcal{C}_0|$ , then  $|\mathcal{C}_Y| \geq 2^\beta$ .  $\square$

For convenience, we define notations in the following for any two codewords  $(\mathbf{v}|\mathbf{w}), (\mathbf{x}|\mathbf{y}) \in \mathcal{C}$ .

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$N(\mathbf{w})$	the number of units of $w$
$N_u(\mathbf{w})$	the number of $u \in w$
$N_{1,1}(\mathbf{w}, \mathbf{y})$	$\#\{i \mid w_i = 1 \text{ or } 1+u, y_i = 1 \text{ or } 1+u, 1 \leq i \leq \beta\}$
$N_{1,u}(\mathbf{w}, \mathbf{y})$	$\#\{i \mid w_i = 1 \text{ or } 1+u, y_i = u, 1 \leq i \leq \beta\}$
$N_{u,1}(\mathbf{w}, \mathbf{y})$	$\#\{i \mid w_i = u, y_i = 1 \text{ or } 1+u, 1 \leq i \leq \beta\}$
$N_s(\mathbf{w}, \mathbf{y})$	$\#\{i \mid w_i = y_i = 1 \text{ or } 1+u, 1 \leq i \leq \beta\}$
$N_d(\mathbf{w}, \mathbf{y})$	$\#\{i \mid w_i = 1, y_i = 1+u \text{ or } w_i = 1+u, y_i = 1, 1 \leq i \leq \beta\}$

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**Lemma 2.6.** *Let  $\mathcal{C}$  be a  $\mathbb{Z}_2R$  linear self-dual code, and  $(\mathbf{v}|\mathbf{w})$ ,  $(\mathbf{x}|\mathbf{y})$  are two code-words in  $\mathcal{C}$ . Then we have  $N_s(\mathbf{w}, \mathbf{y}) \equiv N_d(\mathbf{w}, \mathbf{y}) \pmod{2}$  and  $N_{1,1}(\mathbf{w}, \mathbf{y})$  is even.*

*Proof.* Let  $(\mathbf{v}|\mathbf{w}) = (v_1, \dots, v_\alpha | w_1, \dots, w_\beta)$ ,  $(\mathbf{x}|\mathbf{y}) = (x_1, \dots, x_\alpha | y_1, \dots, y_\beta)$ . In the following, we consider the inner product of  $\langle (\mathbf{v}|\mathbf{w}), (\mathbf{x}|\mathbf{y}) \rangle$ . By (1.3), one has

$$\begin{aligned}
& \langle (\mathbf{v}|\mathbf{w}), (\mathbf{x}|\mathbf{y}) \rangle \\
&= u \sum_{i=1}^{\alpha} v_i x_i + u N_{1,u}(\mathbf{w}, \mathbf{y}) + u N_{u,1}(\mathbf{w}, \mathbf{y}) + N_s(\mathbf{w}, \mathbf{y}) + N_d(\mathbf{w}, \mathbf{y}) + u N_d(\mathbf{w}, \mathbf{y}) \\
&= u \left( \sum_{i=1}^{\alpha} v_i x_i + N_{1,u}(\mathbf{w}, \mathbf{y}) + N_{u,1}(\mathbf{w}, \mathbf{y}) + N_d(\mathbf{w}, \mathbf{y}) \right) + N_s(\mathbf{w}, \mathbf{y}) + N_d(\mathbf{w}, \mathbf{y}) \\
&= 0 \in R.
\end{aligned}$$

Then, we have  $N_s(\mathbf{w}, \mathbf{y}) \equiv N_d(\mathbf{w}, \mathbf{y}) \pmod{2}$ , which implies  $N_{1,1}(\mathbf{w}, \mathbf{y})$  is even.  $\square$

**Definition 2.7.** *Let  $\mathcal{C}$  be a  $\mathbb{Z}_2R$  linear code. If  $\mathcal{C} = \mathcal{C}_X \times \mathcal{C}_Y$ , then  $\mathcal{C}$  is called separable.*

By Theorem 1.2, if  $\mathcal{C}$  is a separable linear code, we have that the generator matrix of  $\mathcal{C}$  in standard form as follows

$$G_s = \left( \begin{array}{cc|ccc} I_\kappa & A & 0 & 0 & 0 \\ 0 & 0 & uB & uI_{\gamma-\kappa} & 0 \\ 0 & 0 & C & D & I_\delta \end{array} \right).$$

The following result is some equivalent conditions of separable  $\mathbb{Z}_2R$  linear self-dual codes.

**Theorem 2.8.** *Let  $\mathcal{C}$  be a  $\mathbb{Z}_2R$  self-dual code of type  $(2\kappa, \beta; \beta + \kappa - 2\delta, \delta; \kappa)$ . Then we have the following statements are equivalent:*

1.  $\mathcal{C}$  is separable.

2.  $\mathcal{C}_X$  is binary self-orthogonal.
3.  $\mathcal{C}_X$  is binary self-dual.
4.  $|\mathcal{C}_X| = 2^\kappa$ .
5.  $\mathcal{C}_Y$  is a self-orthogonal code.
6.  $\mathcal{C}_Y$  is a self-dual code.
7.  $|\mathcal{C}_Y| = 2^\beta$ .

*Proof.* By the similar proof in [9, Theorem 3], we are done.  $\square$

Note that for any vectors  $\mathbf{x}, \mathbf{y} \in \mathcal{C}_X$ , we have

$$wt_H(\mathbf{x} + \mathbf{y}) = wt_H(\mathbf{x}) + wt_H(\mathbf{y}) - 2wt_H(\mathbf{x} * \mathbf{y}),$$

where  $\mathbf{x} * \mathbf{y}$  is the componentwise product of  $\mathbf{x}$  and  $\mathbf{y}$ . If  $wt_H(\mathbf{x}), wt_H(\mathbf{y}), wt_H(\mathbf{x} + \mathbf{y})$  are all doubly-even, then  $wt_H(\mathbf{x} * \mathbf{y}) \equiv 0 \pmod{2}$ , i.e.  $\mathbf{x}$  and  $\mathbf{y}$  are orthogonal. Therefore, by Theorem 2.8, we have below result.

**Corollary 2.9.** *Let  $\mathcal{C}$  be a  $\mathbb{Z}_2R$  linear self-dual code. If  $\mathcal{C}_X$  has all weights doubly-even, then  $\mathcal{C}$  is separable.*

**Corollary 2.10.** *Let  $\mathcal{C}$  be a  $\mathbb{Z}_2R$  linear self-dual code of type  $(\alpha, \beta; \gamma, \delta; \kappa)$ . If  $\delta = 0$ , then  $\mathcal{C}$  is separable.*

*Proof.* If  $\delta = 0$ , then we have for any  $(\mathbf{v}|\mathbf{w}) \in \mathcal{C}$ ,  $\mathbf{w}$  does not contain unit. Thus,  $\mathcal{C}_Y$  is self-orthogonal, by Theorem 2.8, we get  $\mathcal{C}$  is separable.  $\square$

The above corollary implies the following result.

**Corollary 2.11.** *Let  $\mathcal{C}$  be a  $\mathbb{Z}_2R$  linear self-dual code of type  $(\alpha, \beta; \gamma, \delta; \kappa)$ . If  $\mathcal{C}$  is non-separable, then  $\delta \geq 1$ .*

**Definition 2.12.** *If a  $\mathbb{Z}_2R$  linear self-dual code has odd weights, then it is said to be Type 0. If it has only even weights, then the code is said to be Type I. If all the codewords have doubly-even weight then it is said to be Type II.*

By Lemma 2.3, we get the following result.

**Theorem 2.13.** *There donot exist  $\mathbb{Z}_2R$  linear self-dual codes of Type 0.*

The followings examples are of Type I and Type II.

**Example 2.14.** (Type I, separable). Let  $\mathcal{C}$  be a code of type  $(2, 1; 2, 0; 1)$  over  $\mathbb{Z}_2R$ , whose generator matrix is

$$\left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & u \end{array} \right)$$

It is easy to see  $\mathcal{C} = \mathcal{D} \times \mathcal{E}$ , where  $\mathcal{D} = \langle (11) \rangle$ ,  $\mathcal{E} = \langle (u) \rangle$ . Thus  $\mathcal{C}$  is separable. Moreover, we have the weight enumerator of this code is

$$W(x, y) = x^4 + 2x^2y^2 + y^4.$$

Hence,  $\mathcal{C}$  is a self-dual code and  $\mathcal{C}$  is of Type I.

**Example 2.15.** (Type I, non-separable). Let  $\mathcal{C}$  be generated by the following matrix

$$\left( \begin{array}{cccc|ccc} 1 & 0 & 1 & 0 & 0 & 0 & u \\ 0 & 1 & 0 & 1 & 0 & 0 & u \\ 0 & 0 & 0 & 0 & 0 & u & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1+u \end{array} \right)$$

Clearly,  $\mathcal{C}$  is a self-dual code of type  $(4, 3; 3, 1; 2)$ . Note that  $(0 \ 1 \ 0 \ 1), (0 \ 0 \ 1 \ 1) \in \mathcal{C}_X$ , we have  $\langle (0 \ 1 \ 0 \ 1), (0 \ 0 \ 1 \ 1) \rangle = 1$ . Thus  $\mathcal{C}_X$  is not self-orthogonal. By Theorem 2.8,  $\mathcal{C}$  is non-separable. Moreover, the weight enumerator of  $\mathcal{C}$  is

$$W(x, y) = x^{10} + 8x^8y^2 + 14x^4y^6 + 8x^2y^8 + y^{10}.$$

Therefore,  $\mathcal{C}$  is of Type I code.

**Example 2.16.** (Type II, separable). Let  $\mathcal{C}$  be generated by

$$\mathcal{G}_X = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Let  $\mathcal{D}$  be generated by

$$\mathcal{G}_Y = \begin{pmatrix} u & u & 0 & 0 \\ u & 0 & u & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

It is easy to check  $\mathcal{C}$  is a self-dual code over  $\mathbb{Z}_2$  and  $\mathcal{D}$  is a self-dual code over  $R$ . Moreover, the weight of each codeword of  $\mathcal{C}$  and  $\mathcal{D}$  is doubly-even, respectively. Consider the following matrix

$$\left( \begin{array}{c|c} \mathcal{G}_X & \mathbf{0} \\ \hline \mathbf{0} & \mathcal{G}_Y \end{array} \right).$$

It generates a self-dual code  $\mathcal{C} \times \mathcal{D}$  over  $\mathbb{Z}_2R$ . Since  $\mathcal{C}$  and  $\mathcal{D}$  are doubly-even codes, then  $\mathcal{C} \times \mathcal{D}$  is of Type II.



**Example 2.17.** (*Type II, non-separable*). Consider the following matrix

$$\left( \begin{array}{cccc|cc} 1 & 0 & 1 & 0 & 0 & u \\ 0 & 1 & 0 & 1 & 0 & u \\ 0 & 0 & 1 & 1 & 1 & 1+u \end{array} \right),$$

which generates a linear code  $\mathcal{C}$  of type  $(4, 2; 2, 1, 2)$  over  $\mathbb{Z}_2R$ . Note that  $\mathcal{C}$  is a self-orthogonal code and  $|\mathcal{C}| = 2^4$ , then  $\mathcal{C}$  is self-dual. The weight enumerator of this code is

$$W(x, y) = x^8 + 14x^4y^4 + y^8.$$

It is easy to see  $\mathcal{C}_Y$  is not self-orthogonal. By Theorem 2.8, we have  $\mathcal{C}$  is non-separable. To sum up,  $\mathcal{C}$  is of Type II code.

From above examples, we obtain the minimal bounds of  $\alpha, \beta$  for different types.

**Theorem 2.18.** Let  $\mathcal{C}$  be a  $\mathbb{Z}_2R$  self-dual code of type  $(\alpha, \beta; \gamma, \delta; \kappa)$  with  $\alpha \cdot \beta > 0$ .

- If  $\mathcal{C}$  is Type I and separable, then  $\alpha \geq 2, \beta \geq 1$ .
- If  $\mathcal{C}$  is Type I and non-separable, then  $\alpha \geq 4, \beta \geq 2$ .
- If  $\mathcal{C}$  is Type II, then  $\alpha \geq 4, \beta \geq 2$ .

*Proof.* If  $\mathcal{C}$  is Type I and separable, then  $\mathcal{C}_X$  is binary self-dual and  $\mathcal{C}_X$  is self-dual over  $R$ . Thus  $\alpha \geq 2, \beta \geq 1$ .

If  $\mathcal{C}$  is Type I and non-separable. By Lemma 2.3, we have  $\mathcal{C}_x$  are even weight. If  $\alpha = 2$ , then  $\mathcal{C}_x$  is binary self-dual. By Theorem 2.8, it is a contradiction. Thus  $\alpha \geq 4$ . From Example 2.17, we have  $\beta \geq 2$ .

Assume  $\mathcal{C}$  is Type II, by Lemma 2.3, we have  $(\mathbf{1}^\alpha, \mathbf{0}^\beta), (\mathbf{0}^\alpha, \mathbf{u}^\beta) \in \mathcal{C}$ . Since  $\mathcal{C}$  is Type II, then  $\alpha \equiv 0 \pmod{4}, \beta \equiv 0 \pmod{2}$ . Note that Example 2.17, then  $\alpha \geq 4, \beta \geq 2$ .  $\square$

### 3 Several constructions of self-dual codes

In this section, we present several kinds of construction methods for self-dual codes over  $\mathbb{Z}_2R$ . The following theorem is the first one that self-dual codes over  $\mathbb{Z}_2R$  are obtained from other self-dual codes over  $\mathbb{Z}_2R$ .

**Theorem 3.1.** Let  $\mathcal{C}$  be a self-dual code with generator matrix  $G = (G_1|G_2)$  of type  $(\alpha, \beta; \gamma, \delta; \kappa)$ , and  $\mathcal{C}'$  be a self-dual code with generator matrix  $G' = (G'_1|G'_2)$  of type  $(\alpha', \beta'; \gamma', \delta'; \kappa')$ . Then

$$\left( \begin{array}{cc|cc} G_1 & 0 & G_2 & 0 \\ 0 & G'_1 & 0 & G'_2 \end{array} \right)$$

generates a self-dual code  $\mathcal{M}$  of type  $(\alpha + \alpha', \beta + \beta'; \gamma + \gamma', \delta + \delta'; \kappa + \kappa')$ . Moreover, we have the weight enumerator of  $\mathcal{M}$  is

$$W_{\mathcal{M}}(X, Y) = W_{\mathcal{C}}(X, Y)W_{\mathcal{C}'}(X, Y).$$

*Proof.* For any codeword  $(\mathbf{v}|\mathbf{w}) \in \mathcal{M}$ , we have

$$(\mathbf{v}|\mathbf{w}) = (A_{1 \times (\gamma + \delta)}, A'_{1 \times (\gamma' + \delta')}) \left( \begin{array}{cc|cc} G_1 & 0 & G_2 & 0 \\ 0 & G'_1 & 0 & G'_2 \end{array} \right),$$

where  $(A_{1 \times (\gamma + \delta)}, A'_{1 \times (\gamma' + \delta')}) \in R^{\gamma + \gamma' + \delta + \delta'}$ . Note that any two rows of generator  $\left( \begin{array}{cc|cc} G_1 & 0 & G_2 & 0 \\ 0 & G'_1 & 0 & G'_2 \end{array} \right)$  are orthogonal, then  $\mathcal{M}$  is a self-orthogonal code. Since  $\mathcal{C}$  and  $\mathcal{C}'$  are self-dual codes, then  $\alpha + 2\beta = 2(\gamma + 2\delta)$  and  $\alpha' + 2\beta' = 2(\gamma' + 2\delta')$ . Note that the length of  $\mathcal{M}$  is  $\alpha + \alpha' + 2\beta + 2\beta'$  and  $|\mathcal{M}| = 2^{\gamma + \gamma' + 2\delta + 2\delta'}$ , then we have  $\mathcal{M}$  is a self-dual code of type  $(\alpha + \alpha', \beta + \beta'; \gamma + \gamma', \delta + \delta'; \kappa + \kappa')$ . It is easy to check that

$$W_{\mathcal{M}}(X, Y) = W_{\mathcal{C}}(X, Y)W_{\mathcal{C}'}(X, Y).$$

□

**Corollary 3.2.** There exist  $\mathbb{Z}_2R$  linear self-dual codes of type  $(\alpha, \beta; \gamma, \delta; \kappa)$  for all even  $\alpha$  and all  $\beta$ .

In the following, we first establish a relationship between  $\mathbb{Z}_2R$  linear self-dual codes and  $\mathbb{Z}_2\mathbb{Z}_4$ -additive self-dual codes. From this relationship, then we can construct self-dual codes over  $\mathbb{Z}_2R$  from self-dual codes over  $\mathbb{Z}_2\mathbb{Z}_4$ , or self-dual codes over  $\mathbb{Z}_2\mathbb{Z}_4$  from self-dual codes over  $\mathbb{Z}_2R$ .

Define a map

$$\begin{aligned} \theta : \mathbb{Z}_2R &\longrightarrow \mathbb{Z}_2\mathbb{Z}_4 \\ (\mathbf{v}|\mathbf{w}) &\longmapsto (\mathbf{v}|\theta(\mathbf{w})), \end{aligned}$$

where  $(\mathbf{v}|\theta(\mathbf{w})) = (\mathbf{v}|\theta(w_1), \dots, \theta(w_\beta))$ ,  $\theta(0) = 0$ ,  $\theta(1) = 1$ ,  $\theta(u) = 2$ ,  $\theta(1 + u) = 3$ .

**Theorem 3.3.** Let  $\mathcal{C}$  be a  $\mathbb{Z}_2R$  linear code with generator matrix  $G$ , and for any two codewords  $(\mathbf{v}|\mathbf{w}), (\mathbf{x}|\mathbf{y}) \in \mathcal{C}$  with  $4 \mid N_{1,1}(\mathbf{w}, \mathbf{y})$ . If  $\mathcal{C}$  is a self-orthogonal code, then  $\theta(\mathcal{C})$  is also a self-orthogonal code over  $\mathbb{Z}_2\mathbb{Z}_4$ . Furthermore, if  $\mathcal{C}$  is a self-dual code, then  $\theta(\mathcal{C})$  is also a self-dual code over  $\mathbb{Z}_2\mathbb{Z}_4$ .

*Proof.* Since  $\mathcal{C}$  is a self-orthogonal code, then for any two codewords  $(\mathbf{v}|\mathbf{w})$  and  $(\mathbf{x}|\mathbf{y}) \in \mathcal{C}$ , we have

$$\langle (\mathbf{v}|\mathbf{w}), (\mathbf{x}|\mathbf{y}) \rangle = u \sum_{i=1}^{\alpha} v_i x_i + \sum_{j=1}^{\beta} w_j y_j = 0 \in R.$$

Since  $4 \mid N_{1,1}(\mathbf{w})$  and  $N_{1,1}(\mathbf{w}, \mathbf{y}) = N_s(\mathbf{w}, \mathbf{y}) + N_d(\mathbf{w}, \mathbf{y})$ , following the notations in Section 2, we get

$$\begin{aligned} & \langle (\mathbf{v}|\mathbf{w}), (\mathbf{w}, \mathbf{y}) \rangle \\ &= u \sum_{i=1}^{\alpha} v_i x_i + u N_{1,u}(\mathbf{w}, \mathbf{y}) + u N_{u,1}(\mathbf{w}, \mathbf{y}) + N_s(\mathbf{w}, \mathbf{y}) + N_d(\mathbf{w}, \mathbf{y}) + u N_d(\mathbf{w}, \mathbf{y}) \\ &= u \left( \sum_{i=1}^{\alpha} v_i x_i + N_{1,u}(\mathbf{w}, \mathbf{y}) + N_{u,1}(\mathbf{w}, \mathbf{y}) + N_d(\mathbf{w}, \mathbf{y}) \right) + N_s(\mathbf{w}, \mathbf{y}) + N_d(\mathbf{w}, \mathbf{y}) \\ &= u \left( \sum_{i=1}^{\alpha} v_i x_i + N_{1,u}(\mathbf{w}, \mathbf{y}) + N_{u,1}(\mathbf{w}, \mathbf{y}) + N_d(\mathbf{w}, \mathbf{y}) \right) = 0 \in R, \end{aligned}$$

which implies that  $2 \mid \left( \sum_{i=1}^{\alpha} v_i x_i + N_{1,u}(\mathbf{w}, \mathbf{y}) + N_{u,1}(\mathbf{w}, \mathbf{y}) + N_d(\mathbf{w}, \mathbf{y}) \right)$ . Note that

$$\begin{aligned} & \langle (\mathbf{v}|\theta(\mathbf{w})), (\mathbf{x}|\theta(\mathbf{y})) \rangle \\ &= 2 \sum_{i=1}^{\alpha} v_i x_i + 2 N_{1,u}(\mathbf{w}, \mathbf{y}) + 2 N_{u,1}(\mathbf{w}, \mathbf{y}) + N_s(\mathbf{w}, \mathbf{y}) + N_d(\mathbf{w}, \mathbf{y}) + 2 N_d(\mathbf{w}, \mathbf{y}) \\ &= 2 \left( \sum_{i=1}^{\alpha} v_i x_i + N_{1,u}(\mathbf{w}, \mathbf{y}) + N_{u,1}(\mathbf{w}, \mathbf{y}) + N_d(\mathbf{w}, \mathbf{y}) \right) + N_s(\mathbf{w}, \mathbf{y}) + N_d(\mathbf{w}, \mathbf{y}) \\ &= 2 \left( \sum_{i=1}^{\alpha} v_i x_i + N_{1,u}(\mathbf{w}, \mathbf{y}) + N_{u,1}(\mathbf{w}, \mathbf{y}) + N_d(\mathbf{w}, \mathbf{y}) \right) + N_{1,1}(\mathbf{w}, \mathbf{y}). \end{aligned}$$

Since  $4 \mid N_{1,1}(\mathbf{w}, \mathbf{y})$  and  $2 \mid \left( \sum_{i=1}^{\alpha} v_i x_i + N_{1,u}(\mathbf{w}, \mathbf{y}) + N_{u,1}(\mathbf{w}, \mathbf{y}) + N_d(\mathbf{w}, \mathbf{y}) \right)$ , then

$$\langle (\mathbf{v}|\theta(\mathbf{w})), (\mathbf{x}|\theta(\mathbf{y})) \rangle = 0 \in \mathbb{Z}_4.$$

Therefore, we have  $\theta(\mathcal{C})$  is a self-orthogonal code over  $\mathbb{Z}_2\mathbb{Z}_4$ .

Furthermore, if  $\mathcal{C}$  is a self-dual code with generator matrix  $G$ , we have  $\langle \theta(\mathcal{C}), \theta(G) \rangle = 0$ . Hence,  $\langle \theta(\mathcal{C}), \langle \theta(G) \rangle \rangle = 0$ , which implies that  $\theta(\mathcal{C}) \subseteq \langle \theta(G) \rangle^\perp$ . Note that  $\theta$  is a bijective, then

$$|\theta(\mathcal{C})| = 2^{\gamma+2\delta}, \quad |\langle \theta(G) \rangle^\perp| = 2^{\alpha+2\beta} / |\langle \theta(G) \rangle| = 2^{\gamma+2\delta}.$$

This implies that  $\theta(\mathcal{C})$  is a  $\mathbb{Z}_2R$  linear code. Together with  $\theta(\mathcal{C})$  is self-orthogonal and  $|\theta(\mathcal{C})| = 2^{\gamma+2\delta}$ , we have  $\theta(\mathcal{C})$  is also self-dual.  $\square$

Similar as proof in Theorem 3.3, we have below result.

**Theorem 3.4.** *Let  $\mathcal{C}$  be a  $\mathbb{Z}_2R$  linear code with generator matrix  $G$ , and for any codewords  $(\mathbf{v}|\mathbf{w}), (\mathbf{x}|\mathbf{y}) \in \mathcal{C}$  such that  $4 \mid N_{1,1}(\mathbf{w}, \mathbf{y})$ . If  $\mathcal{C}$  is a self-orthogonal code, then  $\theta^{-1}(\mathcal{C})$  is also a self-orthogonal code over  $\mathbb{Z}_2R$ . Furthermore, if  $\mathcal{C}$  is a self-dual code, then  $\theta^{-1}(\mathcal{C})$  is also a self-dual code over  $\mathbb{Z}_2R$ .*

For convience, we Let  $\mathcal{C}$  be a self-dual code of length  $\ell = \alpha + \beta$  and  $(G_0|G_1) = (\mathbf{g}_i|\mathbf{r}_i)$  be the generator matrix of  $\mathcal{C}$ , where  $\mathbf{g}_i(\mathbf{r}_i)$  is the  $i$ -th row of  $G_0(G_1)$ ,  $1 \leq i \leq \gamma + \delta$ , respectively. In the following, we use a building-up approach to construct self-dual codes of different lengths. Here, we just give the proof of Theorem 3.5, since Theorem 3.6 and Theorem 3.7 can be proved by similar way.

**Theorem 3.5.** *Assume the notations given as above. Let  $\mathbf{x}$  be a vector in  $\mathbb{Z}_2^\alpha$  with odd  $wt_H(\mathbf{x})$  and  $\mathbf{y}$  be a vector in  $u\mathbb{Z}_2^\beta$  with  $\langle \mathbf{r}_i, \mathbf{y} \rangle = 0$  for  $1 \leq i \leq \gamma + \delta$ . Suppose that  $h_i = \langle \mathbf{g}_i, \mathbf{x} \rangle$  for  $1 \leq i \leq \gamma + \delta$ . Then the following matrix*

$$G = \left( \begin{array}{ccc|c} 1 & 0 & \mathbf{x} & \mathbf{y} \\ \hline h_1 & h_1 & \mathbf{g}_1 & \mathbf{r}_1 \\ \vdots & \vdots & \vdots & \vdots \\ h_{\gamma+\delta} & h_{\gamma+\delta} & \mathbf{g}_{\gamma+\delta} & \mathbf{r}_{\gamma+\delta} \end{array} \right)$$

*generates a self-dual code  $\mathcal{D}$  over  $\mathbb{Z}_2R$  of length  $\ell + 2$ . Moreover, we have  $\mathcal{C}$  is separable if and only if  $\mathcal{D}$  is separable.*

*Proof.* It is easy to obtain

$$GG^T = 0.$$

Hence,  $\mathcal{D}$  is self-orthogonal. Since  $|\mathcal{D}| \cdot |\mathcal{D}^\perp| = 2^{\alpha+2+2\beta}$ , so  $\mathcal{D}$  is self-dual if and only if

$$|\mathcal{D}| = 2^{\frac{\alpha+2+2\beta}{2}}.$$

Now, we begin computing the size of code  $\mathcal{D}$ . We first assume that there exist two vectors  $(\mathbf{v}_1|\mathbf{w}_1)$  and  $(\mathbf{v}_2|\mathbf{w}_2)$  such that

$$(\mathbf{v}_1|\mathbf{w}_1)G = (\mathbf{v}_2|\mathbf{w}_2)G, \text{ where } \mathbf{v}_i \in \mathbb{Z}_2, \mathbf{w}_i \in R^{\gamma+\delta}, i = 1, 2,$$

i.e.

$$(\mathbf{v}|\mathbf{w})G = 0, \text{ where } \mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2, \mathbf{w} = \mathbf{w}_1 - \mathbf{w}_2.$$

Then

$$\mathbf{v}(1 \ 0 \ \mathbf{x} \mid \mathbf{y}) + \mathbf{w} \left( \begin{array}{ccc|c} h_1 & h_1 & \mathbf{g}_1 & \mathbf{r}_1 \\ \vdots & \vdots & \vdots & \vdots \\ h_{\gamma+\delta} & h_{\gamma+\delta} & \mathbf{g}_{\gamma+\delta} & \mathbf{r}_{\gamma+\delta} \end{array} \right) = 0.$$

We have

$$\mathbf{v} + \mathbf{w} \begin{pmatrix} h_1 \\ \vdots \\ h_{\gamma+\delta} \end{pmatrix} = 0 \text{ and } \mathbf{w} \begin{pmatrix} h_1 \\ \vdots \\ h_{\gamma+\delta} \end{pmatrix} = 0.$$

This implies that  $\mathbf{v} = 0$ . Hence, we have

$$\mathbf{w} \left( \begin{array}{c|c} g_1 & r_1 \\ \vdots & \vdots \\ g_{\gamma+\delta} & r_{\gamma+\delta} \end{array} \right) = 0. \quad (3.1)$$

Note that

$$\left( \begin{array}{c|c} \mathbf{g}_1 & \mathbf{r}_1 \\ \vdots & \vdots \\ \mathbf{g}_{\gamma+\delta} & \mathbf{r}_{\gamma+\delta} \end{array} \right)$$

is the generator matrix of  $\mathcal{C}$ , then from equation (3.1), we have  $\mathbf{w} = 0$ . This shows that

$$\mathcal{D} = 2^{\gamma+1+2\delta}.$$

Note that  $\mathcal{C}$  is self-dual, then  $\alpha + 2\beta = 2(\gamma + \delta)$ . Together with  $|\mathcal{D}| = 2^{\frac{\alpha+2+2\beta}{2}}$ , we obtain  $\mathcal{D}$  is self-dual.

It is easy to see  $\mathbf{y}$  is orthogonal to  $\mathbf{y}$  and  $\mathbf{r}_i$ , where  $1 \leq i \leq \gamma + \delta$ . Therefore, by Theorem 2.8, we have  $\mathcal{D}$  is separable if and only if  $\mathcal{C}$  is separable.  $\square$

**Theorem 3.6.** *Assume the notations given as above. Let  $\mathbf{y}$  be a vector in  $R^\beta$  with odd  $wt_L(\mathbf{y})$  and  $\mathbf{x}$  be a vector in  $\mathbb{Z}_2^\alpha$  such that  $wt_H(x)$  is even and  $\langle \mathbf{g}_i, \mathbf{x} \rangle = 0$  for  $1 \leq i \leq \gamma + \delta$ . Let  $t$  be a unit in  $R$  and  $s_i = \langle \mathbf{r}_i, \mathbf{y} \rangle$  for  $1 \leq i \leq \gamma + \delta$ . Then the following matrix*

$$G = \left( \begin{array}{c|ccc} \mathbf{x} & 1 & 0 & \mathbf{y} \\ \hline \mathbf{g}_1 & s_1 & ts_1 & \mathbf{r}_1 \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{g}_{\gamma+\delta} & s_{\gamma+\delta} & ts_{\gamma+\delta} & \mathbf{r}_{\gamma+\delta} \end{array} \right)$$

*generates a self-dual code  $\mathcal{E}$  over  $\mathbb{Z}_2 R$  of length  $m + 2$ . Moreover, we have  $\mathcal{E}$  is separable if and only if  $\mathcal{C}$  is separable.*

**Theorem 3.7.** Assume the notations given as above. Let  $\mathbf{x}$  be a vector in  $\mathbb{Z}_2^\alpha$  with odd  $wt_H(\mathbf{x})$  and  $\mathbf{y}$  be a vector in  $R^\beta$  with odd  $wt_L(\mathbf{y})$ . Let  $\mathbf{e}$  be a vector in  $\mathbb{Z}_2^\alpha$  such that  $wt_H(\mathbf{e})$  is even and  $\langle \mathbf{g}_i, \mathbf{e} \rangle = 0$  for  $1 \leq i \leq \gamma + \delta$ , and  $\mathbf{a}$  be a vector in  $u\mathbb{Z}_2^\beta$  satisfying  $\langle \mathbf{r}_i, \mathbf{a} \rangle = 0$  for  $1 \leq i \leq \gamma + \delta$ . Suppose that  $\langle (\mathbf{x}|\mathbf{a}), (\mathbf{e}|\mathbf{y}) \rangle = 0$ ,  $h_i = \langle \mathbf{g}_i, \mathbf{x} \rangle$ ,  $s_i = \langle \mathbf{r}_i, \mathbf{y} \rangle$  for  $1 \leq i \leq \gamma + \delta$ . Let  $t$  be a unit in  $R$ , then the following matrix

$$G = \left( \begin{array}{ccc|ccc} 1 & 0 & \mathbf{x} & 0 & 0 & \mathbf{a} \\ 0 & 0 & \mathbf{e} & 1 & 0 & \mathbf{y} \\ \hline h_1 & h_1 & \mathbf{g}_1 & s_1 & ts_1 & \mathbf{r}_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ h_{\gamma+\delta} & h_{\gamma+\delta} & \mathbf{g}_{\gamma+\delta} & s_{\gamma+\delta} & ts_{\gamma+\delta} & \mathbf{r}_{\gamma+\delta} \end{array} \right)$$

generates a self-dual code  $\mathcal{F}$  over  $\mathbb{Z}_2 R$  of length  $m + 4$ . Furthermore, if  $\langle \mathbf{x}, \mathbf{e} \rangle = 0 \in \mathbb{Z}_2$ , we have  $\mathcal{F}$  is separable if and only if  $\mathcal{C}$  is separable. Otherwise, we have  $\mathcal{F}$  is non-separable.

## 4 The structure of two-weight self-dual codes

Note that if  $\mathcal{C}$  is self-dual, then  $(\mathbf{1}^\alpha | \mathbf{0}^\beta), (\mathbf{0}^\alpha | \mathbf{u}^\beta) \in \mathcal{C}$ , where  $\alpha \cdot \beta \neq 0$ . This implies that  $\mathcal{C}$  has at least two different weights. So, we study a class of self-dual codes with two different weights and get the structure of these self-dual codes. Now, we assume that  $\mathcal{C}$  is a two-weight linear code over  $\mathbb{Z}_2 R$  with  $\alpha \cdot \beta \neq 0$ , and  $(\mathbf{1}^\alpha | \mathbf{0}^\beta), (\mathbf{0}^\alpha | \mathbf{u}^\beta) \in \mathcal{C}$ . Since  $(\mathbf{1}^\alpha | \mathbf{0}^\beta), (\mathbf{0}^\alpha | \mathbf{u}^\beta)$  and  $(\mathbf{1}^\alpha | \mathbf{u}^\beta)$  are all in  $\mathcal{C}$ , then  $\alpha = 2\beta$ . Note that  $\alpha + 2\beta = n$ , then  $\alpha = n/2, \beta = n/4$ . Note that if  $4 \nmid n$ , these codes do not exist. Thus, we assume  $4 \mid n$  in this section.

**Lemma 4.1.** Assume the notations given as above. Then the weight distribution of linear code  $\mathcal{C}$  is obtained as follows

Weight	Frenquence
0	1
$n$	1
$\frac{n}{2}$	$ \mathcal{C}  - 2$

We define a linear subcode of  $\mathcal{C}$  as follows

$$\mathcal{C}^* = \{(\mathbf{0}^\alpha | \mathbf{0}^\beta), (\mathbf{1}^\alpha | \mathbf{0}^\beta)(\mathbf{0}^\alpha | \mathbf{u}^\beta)(\mathbf{1}^\alpha | \mathbf{u}^\beta)\}.$$

**Lemma 4.2.** *Let  $\mathcal{C}$  be a two-weight linear code. For any codeword  $(\mathbf{v}|\mathbf{w}) \in \mathcal{C} \setminus \mathcal{C}^*$ , we have*

- $N(\mathbf{w}) = \frac{n}{4}$ ,  $N_u(\mathbf{w}) = 0$ ,  $wt_H(\mathbf{v}) = \frac{n}{4}$ ; or
- $N(\mathbf{w}) = 0$ ,  $N_u(\mathbf{w}) = \frac{n}{8}$ ,  $wt_H(\mathbf{v}) = \frac{n}{4}$ .

*Proof.* If  $N(\mathbf{w}) \neq 0$ , then

$$wt_L(u(\mathbf{v}|\mathbf{w})) = wt_L(u\mathbf{w}) = 2N(\mathbf{w}) = \frac{n}{2}.$$

Thus,  $N(\mathbf{w}) = \frac{n}{4}$ . Note that  $wt_L(\mathbf{v}|\mathbf{w}) = \frac{n}{2}$  and  $\beta = \frac{n}{4}$ , we have  $wt_H(\mathbf{v}) = \frac{n}{4}$ ,  $N_u(\mathbf{w}) = 0$ .

If  $N(\mathbf{w}) = 0$ , then  $N_u(\mathbf{w}) \neq 0$ . Note that  $(\mathbf{0}^\alpha|\mathbf{u}^\beta) \in \mathcal{C}$ , then

$$wt_L((\mathbf{v}|\mathbf{w}) + (\mathbf{0}^\alpha|\mathbf{u}^\beta)) = \frac{n}{2}.$$

Thus  $N_u(\mathbf{w}) = \frac{n}{8}$ ,  $wt_H(\mathbf{v}) = \frac{n}{4}$ . □

**Example 4.3.** *Let  $\mathcal{C}_1$  be generated by  $\left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & u \end{array} \right)$ , then*

$$\mathcal{C}_1 = \{(00|0), (11|0), (00|u), (11|u)\}.$$

*It is easy to check that  $\mathcal{C}_1$  is a self-dual code with two nonzero weights.*

*Let  $\mathcal{C}_2$  be generated by  $\left( \begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 0 & 1 \end{array} \right)$ , then*

$$\mathcal{C}_2 = \{(00|0), (11|0), (10|1), (01|1), (10|1), (10|1+u), (01|1), (01|1+u)\}.$$

*It is a two-weight linear code, which confirms the result in Lemma 4.2.*

By above discussion, we get the maximal bound of  $\delta$ .

**Lemma 4.4.** *Assume the notations given as above, then  $\delta \leq 1$ .*

*Proof.* We assume that  $(\mathbf{v}|\mathbf{w}) \in \mathcal{C}$  with  $N(\mathbf{w}) \neq 0$ . By Lemma 4.2, we have  $N(\mathbf{w}) = \frac{n}{4}$ . Since  $\beta = \frac{n}{4}$ , then we get  $\delta \leq 1$ . □

With above preparation we can obtain the main result in this section.

**Theorem 4.5.** *Let the notations be given as above. Then  $\mathcal{C}$  is self-dual if and only if the generator matrix of  $\mathcal{C}$  is permutation equivalent to*

$$\left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & u \end{array} \right) \text{ or } \left( \begin{array}{cccc|cc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & u \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array} \right).$$

*Proof.* Let  $G$  be the generator matrix of  $\mathcal{C}$ . By Lemma 4.1, we have the weight enumerator of  $\mathcal{C}$  is

$$W_{\mathcal{C}}(X, Y) = X^n + (|\mathcal{C}| - 2)X^{\frac{n}{2}}Y^{\frac{n}{2}} + Y^n. \quad (4.1)$$

By Theorem 1.6, the weight enumerator of  $\mathcal{C}^\perp$  satisfies

$$\begin{aligned} & |\mathcal{C}|W_{\mathcal{C}^\perp}(X, Y) = W_{\mathcal{C}}(X + Y, X - Y) \\ &= (X + Y)^n + (|\mathcal{C}| - 2)(X^2 - Y^2)^{\frac{n}{2}} + (X - Y)^n \\ &= X^n + C_n^1 X^{n-1}Y + C_n^2 X^{n-2}Y^2 + \dots + C_n^{n-1}XY^{n-1} + Y^n \\ &\quad + (|\mathcal{C}| - 2) \left( X^n - C_n^{\frac{1}{2}}(X^2)^{\frac{n}{2}-1}Y^2 + \dots + Y^n \right) \\ &\quad + X^n - C_n^1 X^{n-1}Y + C_n^2 X^{n-2}Y^2 + \dots - C_n^{n-1}XY^{n-1} + Y^n \\ &= -(|\mathcal{C}| - 2) \left( C_n^{\frac{1}{2}}(X^2)^{\frac{n}{2}-1}Y^2 - C_n^2(X^2)^{\frac{n}{2}-2}Y^4 + \dots + C_n^{\frac{n}{2}-1}X^2(Y^2)^{\frac{n}{2}-1} \right) \\ &\quad + |\mathcal{C}|(X^n + Y^n) + 2C_n^2 X^{n-2}Y^2 + \dots + 2C_n^{n-2}X^2Y^{n-2} \\ &= |\mathcal{C}|X^n + \left( 2C_n^2 - (|\mathcal{C}| - 2)C_n^{\frac{1}{2}} \right) X^{n-2}Y^2 + \dots \\ &\quad + \left( 2C_n^2 - (|\mathcal{C}| - 2)C_n^{\frac{n}{2}-1} \right) X^2Y^{n-2} + |\mathcal{C}|Y^n. \end{aligned} \quad (4.2)$$

If  $\mathcal{C}$  is self-dual, then  $W_{\mathcal{C}^\perp}(X, Y) = W_{\mathcal{C}}(X, Y)$ . Comparing the coefficients of both sides of equation (4.2), we discuss it in two cases.

i) If  $n = 4$ , it is easy to check

$$W_{\mathcal{C}^\perp}(X, Y) = W_{\mathcal{C}}(X, Y) = X^4 + 2X^2Y^2 + Y^4.$$

Note that  $\alpha = \frac{n}{2}$ ,  $\beta = \frac{n}{4}$ , we have  $\alpha = 2$ ,  $\beta = 1$ . Since  $|\mathcal{C}| = 2^{\frac{n}{2}} = 4$ , and  $(\mathbf{1}^\alpha | \mathbf{0}^\beta), (\mathbf{0}^\alpha | \mathbf{u}^\beta) \in \mathcal{C}$ , thus  $G$  is permutation equivalent to

$$\mathcal{C} = \left\langle \left( \begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & u \end{array} \right) \right\rangle.$$

ii) If  $n > 4$ , then the right side of equation (4.2) has at least five terms. The second term of right side of equation (4.2)  $\left( 2C_n^2 - (|\mathcal{C}| - 2)C_n^{\frac{1}{2}} \right) X^{n-2}Y^2$  is equal to zero. Hence,  $n = 8$ . Simplifying equation (4.2), one has

$$W_{\mathcal{C}^\perp}(X, Y) = X^8 + 14X^4Y^4 + Y^8.$$



Since

$$W_{\mathcal{C}}(X, Y) = X^8 + (|\mathcal{C}| - 2)X^4Y^4 + Y^8,$$

and  $|\mathcal{C}| = 2^{\frac{n}{2}} = 16$ , then

$$W_{\mathcal{C}^\perp}(X, Y) = W_{\mathcal{C}}(X, Y).$$

This implies when  $n = 8$ , there exists a self-dual code over  $\mathbb{Z}_2R$ . Note that  $\mathcal{C}$  is Type II, by Theorem 2.18, we get  $\alpha = 4$ ,  $\beta = 2$ . Thus,  $\mathcal{C}$  is non-separable. Otherwise,  $\mathcal{C}$  is contradiction with two non-zero weights. By Lemma 4.4, we have  $\delta \leq 1$ . If  $\delta = 0$ , by Corollary 2.10, it is a contradiction. Hence,  $\delta = 1$ ,  $\gamma = 2$ . Let

$$G = \left( \begin{array}{c|c} \mathbf{v}_1 & \mathbf{w}_1 \\ \mathbf{v}_2 & \mathbf{w}_2 \\ \mathbf{v}_3 & \mathbf{w}_3 \end{array} \right)$$

, where  $\mathbf{v}_i \in \mathbb{Z}_2^4$  for  $1 \leq i \leq 3$ ,  $\mathbf{w}_1, \mathbf{w}_2 \in \{0, u\}^2$ ,  $\mathbf{w}_3 \in R^2$ . By Lemma 4.2, we have  $N(\mathbf{w}_3) = 2$ ,  $wt_H(\mathbf{v}_3) = 2$ , i.e.  $\mathbf{w}_3 = (11)$ . Now we fix the sequence of  $(\mathbf{v}_3|\mathbf{w}_3) = (0011|11)$ . Note that  $(\mathbf{1}^4|\mathbf{0}^2) \in \mathcal{C}$ , then we let  $(\mathbf{v}_1|\mathbf{w}_1) = (\mathbf{1}^4|\mathbf{0}^2)$ . Finally, we need to choose  $(\mathbf{v}_2|\mathbf{w}_2)$ . Since  $(\mathbf{v}_2|\mathbf{w}_2) \notin \mathcal{C}^*$ , then by Lemma 4.2, we have  $N_u(\mathbf{w}_2) = 1$ ,  $wt_H(\mathbf{v}_2) = 2$ . Since  $\mathcal{C}$  is non-separable, then  $wt_H(\mathbf{v}_2 * \mathbf{v}_3)$  is odd. To sum up, we get  $G$  is permutation equivalent to

$$\left( \begin{array}{cccc|cc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & u \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array} \right)$$

Conversely, it is easy to check. We finish the proof.  $\square$

From equation (4.2), we get the bound of the minimal distance of dual code  $\mathcal{C}^\perp$ .

**Theorem 4.6.** *Let the notations be given as above. If  $n = \frac{|\mathcal{C}|}{2}$ , then the minimal Hamming distance of  $\mathcal{C}^\perp$  is 4. Otherwise, the minimal Hamming distance of  $\mathcal{C}^\perp$  is 2.*

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